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Random manifolds in the high-temperature expansion of spin and gauge models

T A Larsson†

Department of Theoretical Physics, Royal Institute of Technology, 100 44 Stockholm, Sweden

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Abstract. We write down explicit expressions for the high-temperature expansion of several spin and gauge models with logarithmic action, defined on certain lattices. Such lattices exist in any dimension for spin models, but in the gauge case their existence is only proved modulo a technical assumption. The partition function of the spin models is a gas of closed loops, weighted by N^L, where N is the number of spin components and L is the total length of the loops. The gauge model is a gas of closed, self-avoiding surfaces, weighted by N^χ, where N is the dimension of the representation and χ is the Euler characteristic.

1. Introduction

According to the universality hypothesis, universal quantities, such as critical exponents and certain combinations of critical amplitudes, should only depend on the overall properties of a model and not on details, such as the kind of d-dimensional lattice the model is defined on, inclusion of next-nearest-neighbour interactions, cubic anisotropy, whether a gauge model is defined by single-plaquette actions or not, etc. Loosely speaking, only the dimensionality of the lattice and the symmetries of the action are relevant. This hypothesis has never been rigorously proven, but it is part of the standard lore of the renormalisation group. Assuming universality, we have considerable freedom in choosing a representative for a universality class. A natural strategy is to choose a model that in some sense is the simplest. Domany *et al* (1981) (henceforth referred to as DMNS) were the first to choose models with logarithmic action, because although the action is complicated, the partition function is polynomial rather than exponential. This model was studied on a honeycomb lattice, where each site is connected to only three others. In other words, the coordination number (CN) of this lattice is three. In that case, a particularly simple expression was given for the general term in the high-temperature (or strong coupling) expansion (HTE).

In a previous paper, this construction was generalised to pure lattice gauge theory in three dimensions (Larsson 1987). Unfortunately, the orthogonality relations were stated incorrectly due to a misconception. Because of this, the technical difficulties became insurmountable, and no formal expression *à la* DMNS was given. This error, and the correct derivation of the partition function, was pointed out by A Maritan. The motivation for this paper is partly to correct our previous work, but also to extend

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it in several directions. First, we have found a simple way to construct lattices with CN = 3 in any dimension d, thus showing that there is an O(N) model in any d which is exactly equivalent to a gas of self-avoiding (SA) loops. Modulo a technical assumption, this construction can be generalised to lattices where at most three elementary $(p+1)$-cells meet at each p-cell, for arbitrary p and d. Hence we can also show that there is a lattice gauge model whose HTE consists of closed SA surfaces. Moreover, the same strategy can also be applied to models where the fluctuating variables live on p-cells and interact around $(p+1)$-cells, at least if the gauge group is Abelian. The case of external sources is also discussed.

2. Explicit construction of lattices with low coordination number

We define the coordination number (CN) of a lattice to be the maximum number of links emanating from any node. A square lattice has CN = 4 and a d-dimensional hypercubic lattice has CN = $2d$. There are several well known examples of two-dimensional lattices with CN = 3, e.g. the honeycomb and brick lattices, which in fact are isomorphic. Obviously, only one-dimensional lattices can have a CN less than three, so all interesting models will be defined on lattices with CN ≥ 3.

The construction that we will generalise is the 4-8 lattice, depicted in figure 1. It can be considered to descend from a square lattice, where each node has been replaced by a diamond as in figure 2. In three dimensions, the analogous lattices are constructed from a cubic lattice, which evidently has CN = 6. Each node is now replaced by an octahedron (double pyramid), which lowers the CN to 5. In order to further lower the CN, we delete some of the links of the octahedron. This must be done in a way which does not lower the effective dimensionality, e.g. by splitting the lattice into infinitely many disconnected two-dimensional lattices. The step from figure 3(b) to figure 3(c) is an admissible reduction. It is easy to see that one can reach any of the links 1 to 6

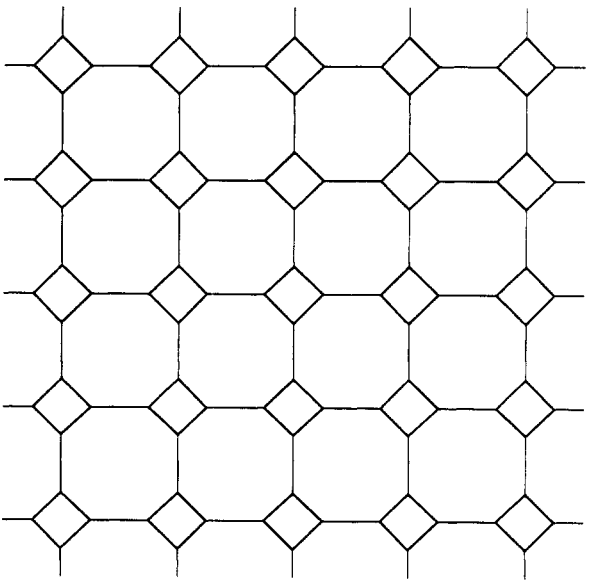

Figure 1. The 4-8 lattice.

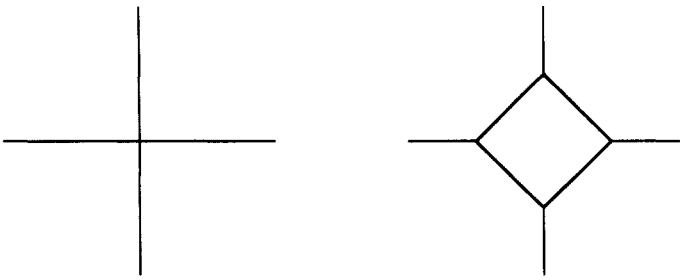

Figure 2. Construction of the 4-8 lattice from the square lattice.

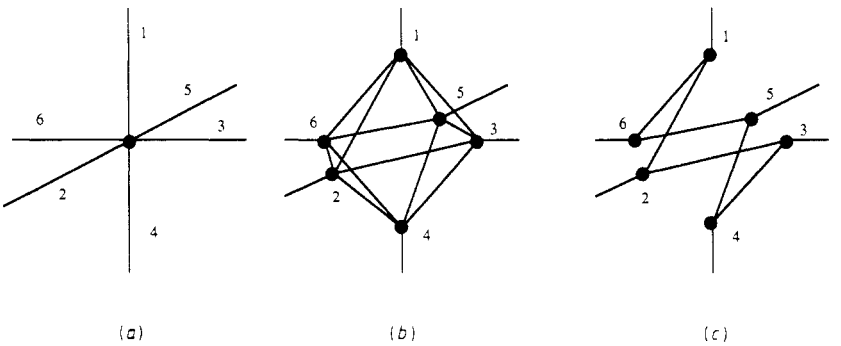

Figure 3. Replacing a node by a structure with CN = 3.

from all others, by following the remaining links in the octahedron. Thus, for an external observer who is not interested in the details of the node, the connectivity of the lattice has not changed. In fact, we can remove one more link without splitting the lattice, e.g. the one between nodes 1 and 6 in figure 3(c). However, this operation does not lower the CN further, since all nodes except two (1 and 6 in this case) are still connected to three other nodes, and it is the maximal connectivity that determines the CN.

This process can now be generalised to arbitrary high dimensions, which is most easily seen as follows. The node in figure 3(a) is topologically equivalent to the diagram in figure 4(a): the incoming links, numbered from 1 to 6, are all connected to it. This node is expanded to the diagram in figure 4(b), where all six nodes are connected to each of the others, except to the one directly opposite to it. Finally, the links inside the hexagon are deleted. Obviously, we can get from each link to any other, by following the sides of the hexagon. In d dimensions exactly the same argument applies, with the hexagon replaced by a $2d$-gon.

The lattice is potentially anisotropic, which could possibly change the universality class of a model defined on it. If each node in the cubic lattice is replaced by figure 3(c), the direction from the closer, lower left-hand corner (triangle 246) to the farther, upper right-hand corner (triangle 135) is singled out. However, this is in fact no problem, since one could single out other directions by rotating figure 3(c). The rotations can be done locally at each node and at random, without changing the model in any way, and thus isotropy can be restored (see figure 4).

In analogy with the CN, we define the second coordination number (CN_2) of a lattice to be the maximal number of plaquettes that are bordered by any link. For a

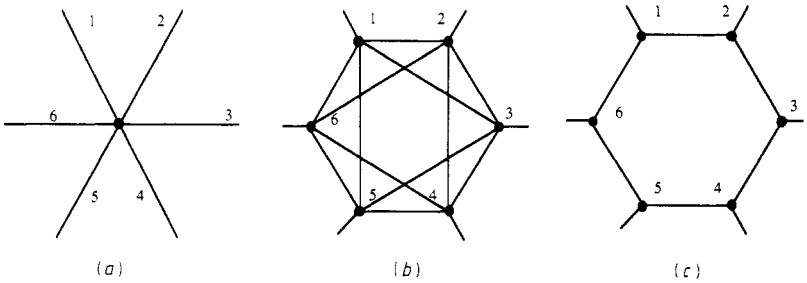

Figure 4. A topologically equivalent representation of figure 3.

hypercubic lattice in d dimensions, $CN_2 = 2(d-1)$. It is obvious that CN_2 is at least three in three or more dimensions, while two-dimensional lattices have $CN_2 = 2$. The pth coordination number CN_p can be defined analogously. A lattice, such that the maximal number of p-cells (elementary cells of dimension p) which border any $(p-1)$-cell is k, has the property that $CN_p = k$.

A three-dimensional lattice with the property that $CN_2 = 3$ was constructed by Larsson (1987), as follows. The two-dimensional brick lattice, which is isomorphic to the honeycomb lattice, is extended in the third direction to make one layer of three-dimensional bricks. A second layer is placed on top of the first one, but displaced as in figure 5, to ensure that no lines are precisely on top of each other. This procedure is then repeated with the layers alternatively parallel to layers one and two. There is a plaquette wherever two bricks meet, and a link wherever two plaquettes meet. This is the three-dimensional brick lattice, and it obviously has $CN_2 = 3$. A four-dimensional brick lattice with $CN_3 = 3$ can analogously be constructed by stacking displaced layers of three-dimensional brick lattices, etc. A d-dimensional brick lattice has $CN_{d-1} = 3$.

The brick lattices solve the problem of finding a lattice with $CN_p = 3$ when $d = p+1$, but not when $d > p+1$. However, the construction in the previous paragraph can in principle be generalised to find new lattices with this desired property. In three dimensions, we replace the plaquettes meeting at a link with figure 2, continued perpendicular to the plane of the paper, as in figure 6. A new problem is encountered here. Close to the corners, the new surfaces do not fit together. Hence, some 'surgery' has to be done to make them fit, but as one can easily verify in figure 6, this does not cause any link to border more than three plaquettes, and CN_2 is still three.

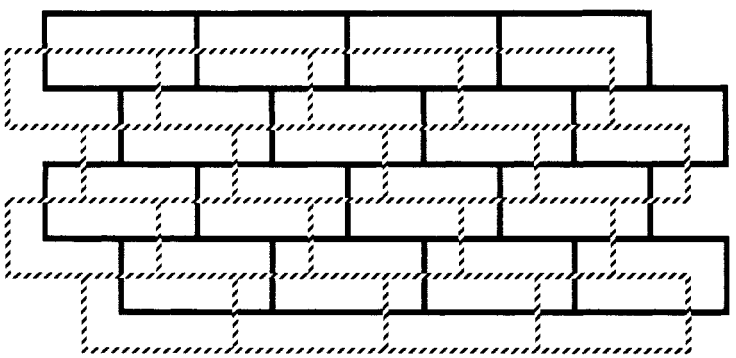

Figure 5. The three-dimensional brick lattice ($CN_2 = 3$).

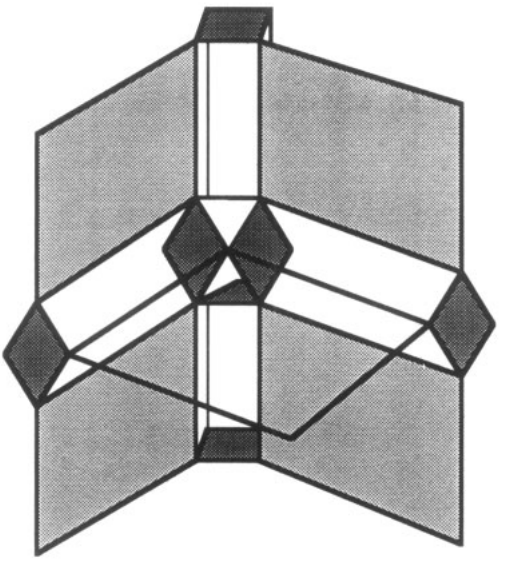

Figure 6. Another three-dimensional lattice with $CN_2 = 3$. Special care is taken to ensure that the plaquettes fit nicely together at the corners.

In four dimensions, we expect that links can be expanded in analogy with figure 3, supplemented by an extra dimension perpendicular to the three shown. Due to the difficulties in visualising this lattice, we have not verified that 'surgery' can be performed to take care of the corners, but we do not expect this to be a serious problem. Modulo such 'surgery' problems, it is immediately clear how to extend our approach to construct lattices with CN_p for any p and d.

3. Spin models on lattices with CN = 3

Assume that a model can be put on a lattice in some natural way by choosing the action to be

$$S = \sum_{\langle ij \rangle} S_{ij} \tag{3.1}$$

where the sum runs over all nearest-neighbour pairs $\langle ij \rangle$ of a standard lattice, such as a d-dimensional hypercubic one. We will refer to this as the standard form of the model. According to our general philosophy, we should be free to modify the action. In particular, we can choose it to be logarithmic

$$S = \sum_{\langle ij \rangle} \ln(1 + S_{ij}) \tag{3.2}$$

where the sites now lie on a d-dimensional lattice with CN = 3. The partition function is simply

$$Z = \mathrm{Tr} \prod_{\langle ij \rangle} [1 + S_{ij}] \tag{3.3}$$

where the Tr operation indicates a sum over all dynamical variables. Z is also a generating function for some class of graphs on the lattice, which constitute the HTE

of this model. Expanding the product, we get a sum of terms where each link $\langle ij \rangle$ contributes either 1 or S_{ij}. In the latter case, we draw a line between i and j, otherwise we leave the link empty. We now turn to some special cases.

3.1. O(N) spin model

Following DMNS, we take the link action to be

$$S_{ij} = Js_i \cdot s_j = Js_i^\alpha s_j^\alpha \tag{3.4}$$

where $s_i = (s_i^\alpha)$ are N-dimensional unit vectors, and the summation convention will be understood henceforth. Because

$$\operatorname{Tr} s^\alpha = \operatorname{Tr} s^\alpha s^\beta s^\gamma = 0 \tag{3.5a}$$

all diagrams that contain a site with one or three emanating links contribute zero to the partition sum. If we choose the measure to be normalised to unity,

$$\operatorname{Tr} 1 = 1. \tag{3.5b}$$

We also have

$$\operatorname{Tr} s^\alpha s^\beta = \frac{1}{N} \delta^{\alpha\beta}. \tag{3.5c}$$

Thus, the only diagrams that contribute will consist of closed loops. For each link we have a factor J/N. Moreover, each loop will contribute $\delta^{\alpha\alpha} = N$. The partition function now becomes

$$Z = \sum_G \left(\frac{J}{N}\right)^{L(G)} N^{C(G)} \tag{3.6}$$

where the sum runs over all configurations G of closed loops, $L(G)$ is the number of links and $C(G)$ is the number of loops (connected components). This is exactly the same formal expression as DMNS arrived at, but it now applies to lattices in three or more dimensions.

Because the action (3.2) is logarithmic, it has severe conceptual problems. For low enough temperatures ($J > 1$) there will be that states that have imaginary action, $S_{ij} < -1$, which usually is considered to be meaningless. However, we will be optimists and hope that useful information can be extracted in spite of this, by some sort of analytic continuation. This has certainly been the case in two dimensions, where this model was instrumental in the calculation of the critical exponents for the O(N) model for general N (Nienhuis 1982, 1988). It should also be noted that the sum (3.6) is well formed. All states with imaginary action have vanished, and all terms in the partition function are positive, for all positive values of J and N.

3.2. Cubic model

The cubic model in N dimensions is given by the action (3.4), but the spins are now constrained to lie on the coordinate axes. In Coxeter's (1963) notation, the spins lie on the corners of an N-dimensional cross polytope, i.e. a square, octaeder, etc...

$$s_i = (0, 0, \ldots, \pm 1, \ldots, 0)$$

where exactly one of the N components is non-zero. We see that the expressions for the products (3.5) are unaltered, and thus it is clear that the partition function will still have the form (3.6).

3.3. U(N) spin models

Instead of considering a spin model that is invariant under O(N) rotations, we can build a theory where the basic variables are unit vectors whose components are complex. These models are invariant under U(N) rotations. The link action is

$$S_{ij} = J(s_i^* \cdot s_j + s_i \cdot s_j^*)$$

where * denotes complex conjugation. Each link now acquires a direction, depending on whether we go to or from the conjugated spin. All combinations of one or three spins give zero when traced over, while (3.5c) is replaced by

$$\mathrm{Tr}\, s^{*\alpha} s^{\beta} = \frac{1}{N}\delta^{\alpha\beta} \qquad \mathrm{Tr}\, s^{\alpha} s^{\beta} = \mathrm{Tr}\, s^{*\alpha} s^{*\beta} = 0.$$

Thus, only configurations of closed, directed loops will survive. The result is (3.6), with the modification that the sum now is taken over all directed loops. Summing over directions gives a factor 2^C, because the directions of the C different loops are independent, and we see that the U(N) model has the same partition function as the O($2N$) model has. Of course, this comes as no surprise, since the models are identical.

3.4. General factorisable spin model

More generally, consider a spin model governed by the action (3.4), but let now the spins take their values in an N-dimensional target manifold M. The trace operation is

$$\mathrm{Tr}\, A(s) = \int A(s)\, \mathrm{d}\mu(s)$$

where $\mathrm{d}\mu(s)$ is the measure on M. By a suitable choice of coordinates on M, one can make the trace satisfy

$$\mathrm{Tr}\, 1 = 1 \qquad \mathrm{Tr}\, s^{\alpha} = 0 \qquad \mathrm{Tr}\, s^{\alpha} s^{\beta} = \lambda^{\alpha}\delta^{\alpha\beta}.$$

This is achieved by rescaling the measure, putting the origin at the centre of mass, and choosing the coordinate system along the main axes of inertia. The triple product, $\mathrm{Tr}\, s^{\alpha} s^{\beta} s^{\gamma}$, is generally different from zero, however. This shows that the HTE of all factorisable models can be chosen to consist of graphs without endpoints, whereas three-point vertices in general cannot be avoided.

3.5. Blöte–Nienhuis O(N) model

Recently Blöte and Nienhuis (1989) introduced a different variant of the O(N) model which also is equivalent to a loop gas. O(N) spins live on the links of a square lattice and interact around plaquettes in the following fashion:

$$Z = \mathrm{Tr} \prod_{ijkl} [1 + u(s_i \cdot s_j + s_j \cdot s_k + s_k \cdot s_l + s_l \cdot s_i) + v(s_i \cdot s_k + s_j \cdot s_l) + w[(s_i \cdot s_j)(s_k \cdot s_l) + (s_j \cdot s_k)(s_l \cdot s_i)]] \tag{3.7}$$

where *ijkl* label the four edges of a plaquette clockwise. Upon expansion Z becomes a sum over closed SA loops:

$$Z = \sum_G \left(\frac{u}{N}\right)^{L_u} \left(\frac{v}{N}\right)^{L_v} \left(\frac{w}{N^2}\right)^{L_w} N^C \tag{3.8}$$

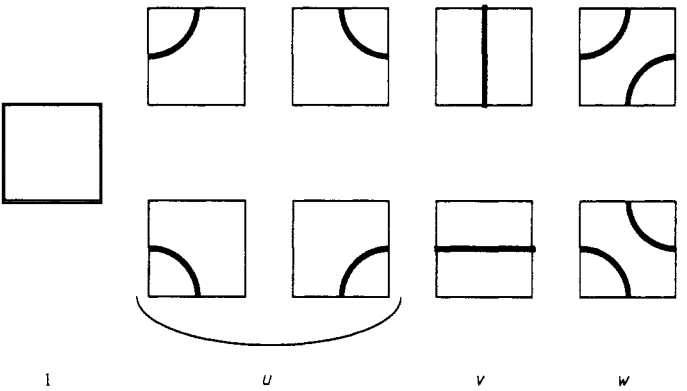

Figure 7. Boltzmann weights of the nine vertices that occur in the Blöte-Nienhuis model (3.6).

where L_x is the number of vertices of type x; see figure 7. In particular, when $u = v = w^{1/2} = J$, (3.8) has the same form as (3.6), with $L = L_u + L_v + 2L_w$, but the set of graphs are different. By combining numerical methods with a mapping of this model onto a solvable nineteen-vertex model, Blöte and Nienhuis identified the universality classes of the critical regions. Here we note that (3.7) can easily be generalised to higher dimensions in such a way that the partition function can still be transformed into a loop gas. For example, in three dimensions, one puts the spins on plaquettes and lets them interact around elementary cubes. For simplicity we limit ourselves to a model where the analogues of $w = 0$, so at most one loop can pass through each elementary cube. Z is then

$$Z = \mathrm{Tr} \prod_{ijklmn} [1 + u(s_i \cdot s_j + 11 \text{ more}) + v(s_i \cdot s_l + 2 \text{ more})].$$

This construction thus also yields an exact equivalence between O(N) spin models and loop gases in any dimension, but as far as we can see this cannot be extended to gauge models.

4. Plaquette models on lattices with $c_{N_2} = 3$

The construction in the previous section can be generalised to lattice models where the basic objects live on links and interact around plaquettes (Larsson 1987). The most important class of this type of model is lattice gauge theory, which has been reviewed by Kogut (1979, 1983) and Drouffe and Zuber (1983).

We assume that the standard form of the action is given on a d-dimensional hypercubic lattice by

$$S = \sum_P S_P \tag{4.1}$$

where the sum runs over all plaquettes P. In complete analogy with what we did for spin models, we define the logarithmic version of this model on a d-dimensional lattice with $c_{N_2} = 3$ by

$$Z = \mathrm{Tr} \prod_P [1 + S_P] \tag{4.2}$$

where the trace Tr now runs over all configurations of link variables. Evidently, (4.2) reduces to the standard action of lattice gauge theory as $S_P \to 0$. Moreover, (4.2) is invariant under all transformations that leave each S_P invariant, so if (4.1) has a gauge symmetry, so does (4.2), and it is appropriate to consider (4.2) as a gauge theory on the lattice.

4.1. O(N) gauge model

The fluctuating variables of this model are orthogonal matrices that belong to the fundamental, N-dimensional representation $O(N)$ and live on the links. We take the plaquette action to be

$$S_P = \mathfrak{B} \operatorname{tr} \prod_{b \in \partial P} O_b = \mathfrak{B} O^{\alpha\beta}_{b_1} O^{\beta\gamma}_{b_2} \ldots O^{\delta\alpha}_{b_k} \tag{4.3}$$

where the product runs over all links b that surround the plaquette P, and tr is the trace in the given representation. Note that tr is different from the ensemble trace Tr. Because $c_{N_2} = 3$, only products of at most three matrices enter in the HTE. The relations analogous to (3.5) are

$$\operatorname{Tr} 1 = 1 \tag{4.4a}$$

$$\operatorname{Tr} O^{\alpha\beta} = 0 \tag{4.4b}$$

$$\operatorname{Tr} O^{\alpha\beta} O^{\gamma\delta} = \frac{1}{N} \delta^{\alpha\gamma} \delta^{\beta\delta} \tag{4.4c}$$

$$\operatorname{Tr} O^{\alpha\beta} O^{\gamma\delta} O^{\varepsilon\varphi} = 0. \tag{4.4d}$$

Equation (4.4c) is the orthogonality relation for the group, and (4.4d) follows because $O(N)$ has a Z_2 centre. Because of (4.4b, d), only graphs where all links border zero or two occupied plaquettes will contribute to the partition function. These graphs form closed SA surfaces. To calculate the fugacity of each such surface, we note that each link contributes a factor N^{-1} by (4.4c). At each site, we have an independent sum over indices, so every site contributes a factor $\delta^{\alpha\alpha} = N$. Finally, each plaquette gives $\mathfrak{B} = (\mathfrak{B}/N)N$ by (4.3). Taken together, we find that the partition function can be written as a sum over closed surfaces G, orientable or not:

$$Z = \sum_G \left(\frac{\mathfrak{B}}{N}\right)^{P(G)} N^{\chi(G)} \tag{4.5}$$

where $P(G)$ is the number of plaquettes (alias the area) and $\chi(G)$ is the Euler characteristic of the surface:

$$\chi(G) = (\#\,\text{sites}) - (\#\,\text{links}) + (\#\,\text{plaquettes}).$$

The Euler characteristic is a topological invariant, which is independent of the particular triangulation of the surface. For closed orientable surfaces, it is given by

$$\chi(G) = 2 \times (\#\,\text{connected components}) - 2 \times (\#\,\text{handles}).$$

It should be noted that the $O(N)$ gauge model does not describe a single SA surface in the $N \to 0$ limit, contrary to the $O(N)$ spin model which describes a single SA walk (a linear polymer in a good solvent). The reason for this well known phenomenon is very easy to see from (4.5). When $N \to 0$ with $\mathfrak{B}/N$ fixed, we can always make a term in (4.5) larger by adding another closed surface with $\chi < 0$. Thus, this limit does not describe a single closed surface, but rather a dense gas of surfaces with many handles (Maritan and Omero 1982).

4.2. U(N) gauge model

The local action (4.3) is changed to

$$S_P = \mathfrak{B}\left[\operatorname{tr} \prod_{b \in \partial P} U_b + \text{cc} \right] \tag{4.6}$$

where the link variables U_b belong to U(N). Each link now has a direction, and the sum over directions is implemented by the complex conjugate term. Only graphs where every link is either traversed once in each direction or not at all have a non-zero weight, because

$$\operatorname{Tr} \bar{U}^{\alpha\beta} U^{\gamma\delta} = \frac{1}{N} \delta^{\alpha\gamma} \delta^{\beta\delta}$$

where $\bar{U}^{\alpha\beta}$ is the complex conjugate of $U^{\alpha\beta}$. In other words, the (α, β)th component of the Hermitian conjugate matrix $\bar{U}$ is $\bar{U}^{\beta\alpha}$. The trace over every other combination, such as

$$U^{\alpha\beta} U^{\gamma\delta} \qquad \bar{U}^{\alpha\beta} U^{\gamma\delta} U^{\varepsilon\varphi} \qquad U^{\alpha\beta} U^{\gamma\delta} U^{\varepsilon\varphi}$$

is zero because of the U(1) centre of U(N). Thus, the graph expansion gives formally the same result (4.5), but the sum now runs over all oriented closed surfaces. On a lattice where all closed surfaces are orientable, such as every three-dimensional lattice, the sum over orientations can be performed and the partition function be expressed as a sum over all unoriented surfaces. Since each connected component can be oriented independently, this means that we get a factor of two for each. Consequently, (4.5) is changed to

$$Z = \sum_G \left(\frac{\mathfrak{B}}{N} \right)^{P(G)} N^{\chi(G)} 2^{C(G)}. \tag{4.7}$$

4.3. General gauge model

The fundamental representations O(N) and U(N) constitute the most simple real and complex irreps of gauge groups, respectively. For a general real irrep R of a gauge group G, we write the action as (4.3), but we now interpret tr as the trace in the appropriate representation. With suitable normalisation, the relations (4.4*a*, *b*, *c*) are automatically satisfied with N being the dimension of the irrep, and it only remains to check (4.4*d*). In group theoretical language (Slansky 1981), this amounts to saying that $R \otimes R \otimes R$ does not contain the singlet or, which is the same thing because of orthogonality between irreps, that $R \otimes R$ does not contain R itself. If $R \otimes R \sim 1 + R + \ldots$, on the other hand, the partition function will contain terms where three plaquettes meet, and hence it will no longer be a sum over proper surfaces.

For a complex representation R, we write the action as (4.6). Orthogonality and normalisation gives

$$\operatorname{Tr} 1 = 1 \tag{4.8a}$$

$$\operatorname{Tr} U^{\alpha\beta} = 0 \tag{4.8b}$$

$$\operatorname{Tr} U^{\alpha\beta} \bar{U}^{\gamma\delta} = \frac{1}{N} \delta^{\alpha\gamma} \delta^{\beta\delta}. \tag{4.8c}$$

Hence, it remains to check that

$$\mathrm{Tr}\; U^{\alpha_1\beta_1} \ldots U^{\alpha_p\beta_p} \bar{U}^{\alpha_{p+1}\beta_{p+1}} \ldots \bar{U}^{\alpha_q\beta_q} = 0 \tag{4.9}$$

for all combinations of p and q such that $0 < p+q \leqslant 3$, except $p = q = 1$. If this is true, which, for example, happens if the group has a U(1) or Z_N centre, $N \geqslant 4$, the partition function is given by (4.5), where the sum runs over all oriented surfaces and N is the dimension of the representation. Group theoretically, (4.9) implies both that $R \otimes R$ does not contain R^*, and that R^* is not equivalent to R. In the first case, the graphs in the HTE contain singular lines where three surfaces meet. In the second case, there will be contributions from graphs where two differently oriented plaquettes are next to each other. Such irreps are known as pseudoreal. For SU(N), $N \geqslant 4$, (4.9) holds because of Z_N invariance. In SU(3), $3 \otimes 3 \sim 3^* + \ldots$. More explicitly, two triple products are non-zero, namely

$$\mathrm{Tr}\; U^{\alpha\beta} U^{\gamma\delta} U^{\varepsilon\varphi} = \mathrm{Tr}\; \bar{U}^{\alpha\beta} \bar{U}^{\gamma\delta} \bar{U}^{\varepsilon\varphi} = \tfrac{1}{6} \varepsilon^{\alpha\gamma\varepsilon} \varepsilon^{\beta\delta\varphi}.$$

The fundamental irrep of SU(2) is pseudoreal ($2 \sim 2^*$), and hence adjacent plaquettes do not need to have the same orientation. Explicitly

$$\mathrm{Tr}\; U^{\alpha\beta} U^{\gamma\delta} = \mathrm{Tr}\; \bar{U}^{\alpha\beta} \bar{U}^{\gamma\delta} = \tfrac{1}{2} \varepsilon^{\alpha\gamma} \varepsilon^{\beta\delta}$$

but all triple products vanish. Thus, the HTE is a sum over closed SA surfaces, orientable or not, but the weights are complicated.

Among the fundamental representations f of the other groups, SO(N) is real and satisfies (4.9), except SO(3) for which $3 \otimes 3$ contains 3^*. Sp($2N$) is pseudoreal but the product of $f \otimes f \otimes f$ does not contain 1, so the partition function is a sum over unoriented graphs with some weights. All exceptional groups except E_7 have $f \otimes f \otimes f \sim 1 + \ldots$, so the graphs contain singular lines. However, our analysis gives that the 56 of E_7 has the same partition function as the fundamental irrep of SO(56), which is quite a non-trivial result.

There is no problem to consider other irreps than the fundamental one. However, if the dynamical variables are chosen to lie in the adjoint representation, the graph expansion always contains singular lines, because the product of an irrep with itself always contains the adjoint. If one works out the Clebsch-Gordan coefficients for the $R \otimes R \otimes R$, one could still be able to show interesting equivalences between models, but we will not attempt this here.

4.4. Discrete gauge groups

Just as O(N) spin models can be approximated by cubic models, we can choose a discrete gauge group on the lattice. As we showed above, the crucial condition is that (4.9) vanishes, which is guaranteed if the group has a Z_2 subgroup. For example, the O(N) gauge model has the same HTE as the discrete subgroup which rotates the cross polytope of subsection 3.2 into itself. This group consists of 2^N reflections and $N!$ relabellings of the coordinate axes (Coxeter 1963). With every group element we can associate a Z_2 charge, which is $+1$ if there is an even number of reflections, and -1 otherwise. This is the sought Z_2 subgroup that ensures that all triple products vanish. It is possible that our observation can lead to more efficient simulation techniques for pure lattice gauge theories, since a discrete group generally is easier to handle numerically than a continuous one. However, it is not clear that the introduction of matter will preserve this equivalence.

4.5. O(N) plaquette model

Following Maritan and Stella (1987), we can take the fluctuating variables to be O(N)-symmetric spins s_b^α that live on the links and interact with the following action:

$$S_P = \mathfrak{B} N^{k/2} \sum_{\alpha=1}^{N} \prod_{b \in \partial P} s_b^\alpha = \mathfrak{B} N^{k/2} \sum_{\alpha=1}^{N} s_{b_1}^\alpha \dots s_{b_k}^\alpha \tag{4.10}$$

where each plaquette P is bounded by k links. This action is clearly somewhat peculiar because the vector index α occurs more than twice when it is summed over. Hence the model does not have a natural tensor structure. It is nevertheless interesting because it will allow us to recover the problem of a single SA random surface in the $N \to 0$ limit.

In the HTE, each occupied plaquette contributes a factor s_b^α to its surrounding links. The variables, being O(N) spins, fulfil the relations (3.5), and hence only closed surfaces survive. The factor N^{-1} from each link is cancelled by the $N^{k/2}$ in the definition of the coupling constant. A plaquette can be considered to be in state α, and from (3.5c) we see that all adjacent plaquettes will be in the same state. For each connected component of the surface, we get an additional sum over α, which gives a factor N. The partition function thus becomes

$$Z = \sum_G \mathfrak{B}^{P(G)} N^{C(G)}. \tag{4.11}$$

It is clear that if the spins are restricted to lie on the coordinate axes, we get a cubic model which has exactly the same HTE as the O(N) plaquette model, for the same reason as in subsection 3.1.

5. Source fields

So far, we have only dealt with spin models in zero magnetic fields and pure gauge models without sources. These models give rise to a HTE that consists of closed manifolds only. When we introduce source (magnetic) fields, this is no longer the case.

Let us first return to the O(N) spin model of subsection 3.1. The simplest way to introduce a magnetic field $H = (H^\alpha)$ is to note that if the standard form of the action is a sum over links $\langle ij \rangle$ and sites k, $S = \Sigma S_{ij} + \Sigma S_k$, the partition function can be approximated for small S.

$$Z = \operatorname{Tr} \exp(\Sigma S_{ij} + \Sigma S_k) = \operatorname{Tr} \exp(\Sigma S_{ij}) \exp(\Sigma S_k) \approx \operatorname{Tr} \Pi(1 + S_{ij}) \Pi(1 + S_k).$$

Thus, we replace the partition function (3.3) by

$$Z = \operatorname{Tr} \prod_{\langle ij \rangle} [1 + J s_i \cdot s_j] \prod_k [1 + H \cdot s_k] \tag{5.1}$$

defined on a d-dimensional lattice with CN = 3. Expanding out (5.1) in graphs, we get an occupied link for each time we choose the second term in the first product, and an occupied site each time we choose the second term in the second product. Clearly, every occupied site contributes a factor $H \cdot s = H^\alpha s^\alpha$. There are now two different ways to obtain an even number of spins at a site to sum over, and thus to get a non-zero contribution to the partition function. Either we have zero or two occupied links leading into an unoccupied site, as before, or we have one or three occupied links leading into an occupied site. We have studied this problem on a general lattice in (Larsson 1985), and we have arrived at a complicated formal expression. This expression can easily be evaluated for any given graph, but it does not give much insight.

The problematic graphs are those containing three links meeting at a site, because they cause the loop to branch and thereby cease to be a manifold. We will evaluate the model (5.1) in the approximation that branch points are ignored. Such an approximation is in fact exact for the Blöte–Nienhuis models considered in section 3.5. Having disposed of three-link vertices, the only new contribution will come from sites where one link meets an occupied site:

$$\operatorname{Tr} H^\alpha s^\alpha s^\beta = H^\alpha \frac{1}{N} \delta^{\alpha\beta} = \frac{H^\beta}{N}. \tag{5.2}$$

The partition function consists of SA graphs containing closed and open strings. The closed ones give the same factor as before. An open string has two end-points, each giving a factor H/N, and links giving a factor J/N. The Kronecker deltas along the string will single out the same component of H at both end-points, and summation over components will give $H^2 = H \cdot H$. Since each open string has two end-points, the partition function is

$$Z = \sum_G \left(\frac{J}{N}\right)^{L(G)} N^{C(G)} \left(\frac{H}{N}\right)^{B(G)} \tag{5.3}$$

where $B(G)$ is the number of end-points and $H = \sqrt{H^2}$.

The gauge models can also be analysed in the approximation that three-plaquette interactions are absent, but here we do not have an example where this is exact. Nevertheless, we define the partition function on a lattice with $\mathrm{CN}_2 = 2$ as

$$Z = \operatorname{Tr} \prod_p [1 + \beta \operatorname{tr} \Pi O] \prod_b [1 + \operatorname{tr} J O_b] \tag{5.4}$$

and the $N \times N$ matrix $J^{\alpha\beta}$ is defined on each link. The new possibility compared with (4.3) is that we have an occupied plaquette next to an occupied link. This yields the factor

$$\operatorname{Tr} J^{\alpha\beta} O^{\alpha\beta} O^{\gamma\delta} = J^{\alpha\beta} \frac{1}{N} \delta^{\alpha\gamma} \delta^{\beta\delta} = \frac{J^{\gamma\delta}}{N} \tag{5.5}$$

for each link on the boundary. Since the normal contribution from a link is $1/N$, the surplus weight is $J^{\gamma\delta}$. It is clear that the occupied links must form unbroken curves that are boundaries of the surfaces, which now are open. The sites along the loop can no longer be traced over independently, which can be accounted for by multiplying each site on the boundary by N^{-1}. Hence, each boundary carries the extra weight

$$N^{-L} J^{\alpha\beta} J^{\beta\gamma} \ldots J^{\delta\alpha} = N^{-L} \operatorname{tr} J^L \tag{5.6}$$

where L is the length (the number of sites) of the boundary. The partition function is thus a sum over open and closed surfaces,

$$Z = \sum_G \left(\frac{\beta}{N}\right)^{P(G)} N^{\chi(G) - L(G)} \prod_B \operatorname{tr} J^{L_B(G)} \tag{5.7}$$

where the product runs over all boundaries B, $L_B(G)$ is the length of B and $L(G) = \Sigma_B L_B$. It should be noted that the Euler index is different for surfaces with holes.

$$\chi(G) = 2(\#\text{ connected components}) - 2(\#\text{ handles}) - (\#\text{ holes}).$$

It should be clear how to generalise this construction to the other kinds of plaquette models defined in section 4.

6. *p*-cell gauge theories

The question now arises if it is possible to construct models whose HTE is given by SA manifolds of dimension $\geqslant 3$. The answer is affirmative and the construction is straightforward when the dynamic variables take values in some Abelian group. Simply put spins on the p-cells of a lattice with $\mathrm{CN}_p = 3$ and define the action as the product of spins around each $(p+1)$-cell. For example, consider a model where Ising spins live on plaquettes on a lattice with $\mathrm{CN}_3 = 3$ and interact around elementary volumes V, with partition function

$$Z = \mathrm{Tr} \prod_V [1 + J s_{p_1} s_{p_2} \ldots s_{p_k}] \tag{6.1}$$

where all $p_i \in \partial V$ and k is the number of plaquettes bordering V. We can express Z as a sum over graphs consisting of closed unoriented 3-manifolds

$$Z = \sum_G J^{V(G)} \tag{6.2}$$

where $V(G)$ is the number of elementary volumes building up G. If the Ising spins in (6.1) are replaced by U(1) spins and a complex conjugate term is added to the action, the sum (6.2) instead runs over oriented 3-manifolds.

The situation is more complicated for non-Abelian quantities because there is no natural way to define a product of ordinary matrices around a p-cell, $p \geqslant 3$. We have recently introduced a new class of models, p-cell lattice gauge models (Larsson 1990), which in principle can be used to build higher-dimensional manifolds in the HTE. The usual gauge models correspond to $p = 1$ whereas the $p = 0$ models are ordinary spin models. We have not been able to evaluate the weight of each graph in the HTE and therefore we only sketch the case $p = 2$ briefly.

Consider a hypercubic lattice where a four-index quantity $U^{\alpha\beta}_{\gamma\delta}(x, ij)$ lives on each plaquette with base point x and direction ij. The indices correspond to the four edges of the plaquette and they can 'hook' to the Us on each of the four neighbours. On the same plaquette with opposite orientation lives an inverse of U, defined by the following conditions:

$$U^{\alpha\beta}_{\gamma\delta}(U^{-1})^{\gamma\delta}_{\varepsilon\varphi} = U^{\gamma\delta}_{\varepsilon\varphi}(U^{-1})^{\alpha\beta}_{\gamma\delta} = U^{\alpha\delta}_{\gamma\varphi}(U^{-1})^{\gamma\beta}_{\varepsilon\delta} = U^{\gamma\beta}_{\varepsilon\delta}(U^{-1})^{\alpha\delta}_{\gamma\varphi} = \delta^{\alpha}_{\varepsilon}\delta^{\beta}_{\varphi}. \tag{6.3}$$

The standard action for the 2-cell gauge model is $S = J \Sigma_{xijk} S_{xijk}$, where $xijk$ denotes an elementary cube with base point x in the ijk direction. The piece of the action associated with a cube in the 123-direction is

$$S_{x123} = J \sum_{\alpha \ldots \tau} [U^{\alpha\beta}_{\gamma\delta}(x, 12) U^{\gamma\varepsilon}_{\phi\kappa}(x, 13) U^{\lambda\sigma}_{\beta\varepsilon}(x, 23)(U^{-1})^{\phi\nu}_{\mu\lambda}(x+3, 12)$$

$$\times (U^{-1})^{\mu\tau}_{\alpha\sigma}(x+2, 13)(U^{-1})^{\delta\kappa}_{\nu\tau}(x+1, 23) + \mathrm{CC}]. \tag{6.4}$$

Note that each U shares an index with each of its four neighbours. Introducing the transpose of U,

$$(U^{\mathrm{t}})^{\alpha\beta}_{\gamma\delta} = U^{\gamma\delta}_{\alpha\beta} \tag{6.5}$$

we can define orthogonal and unitary four-index quantities and the corresponding restrictions of the model (6.4).

Let us introduce a more streamlined notation by considering U as an element in $G \otimes G$, where G is a matrix group: $U = S \otimes T$. The inverse is defined by $U^{-1} = S^{-1} \otimes T^{-1}$, the transpose by $U^t = S^t \otimes T^t$, and the condition (6.3) reads

$$(S \otimes T)(S^{-1} \otimes T^{-1}) = (S^{-1} \otimes T^{-1})(S \otimes T) = (S \otimes T^t)(S^{-1} \otimes (T^t)^{-1})$$
$$= ((S^t)^{-1} \otimes T^{-1})(S^t \otimes T) = I \otimes I.$$

This is important because an invariant measure is inherited from the measure on G; let us denote it by Tr:

$$\begin{aligned} &\mathrm{Tr}\ 1 = 1 \qquad \mathrm{Tr}\ U = \mathrm{Tr}\ U \otimes U \otimes U = 0 \\ &\mathrm{Tr}\ U \otimes U^{-1} = \mathrm{Tr}\ U^{-1} \otimes U = N^{-2} I \otimes I \otimes I \otimes I \end{aligned} \tag{6.6}$$

where N is the dimension of the representation of G. The factor N^{-2} in the last equation is fixed by the conditions $UU^{-1} = I \otimes I$, $\mathrm{tr}\ I \otimes I = N^2$.

It is now clear that if the hypercubic lattice had $c_{N_2} = 3$, the log version of this model would give rise to a HTE consisting of closed SA 3-manifolds. However, this is not the case, and some modifications have to be made to adapt this model to lattices with $c_{N_2} = 3$, because not every plaquette is square on such lattices. There is only one possibility: let the number of indices depend on the shape of the plaquette such that there is always one index for each side. If a plaquette is bordered by an even number of sides, $2K$ say, the corresponding variable takes values in $G^{\otimes K}$, where diagonally opposite links belong to the same G factor. A natural definition of the inverse, transpose and invariant measure is inherited from G. The latter satisfies

$$\mathrm{Tr}\ U \otimes U^{-1} = N^{-K} I^{\otimes 2K}.$$

In principle it is now clear how to evaluate the weight of each graph in the HTE of these models, but in practice this task appears to be very difficult. On lattices where there are plaquettes with an odd number of boundary links, we cannot appeal to a tensor product structure and the situation becomes problematic even in principle.

7. Conclusion and further remarks

We have found that the HTE of many models with logarithmic action is equivalent to a gas of closed SA manifolds. This is of course not the first time the connection between statistical mechanics models and random manifolds is made; on the contrary, many of our references make this observation. The new point is that it suffices to consider proper manifolds without intersections to find models in the right universality class, assuming, of course, that the log models on our rather pathological lattices have the same universal behaviour as the corresponding standard model.

There have been problems to simulate random surfaces on computers. It is energetically favourable for the surface to develop 'fingers', i.e. very narrow tubes, and hence the surface degenerates to essentially a branched polymer, or lattice animal (David 1986). To our knowledge most simulations have been carried out on connected surfaces, and it is possible that the existence of other connected components stabilises the surface. Smaller components inside a large one may prevent the latter from collapsing, since the SA constraint makes the manifolds 'hard'. Anyway, some of the random surface models constructed in this paper should definitely not be in the lattice animal universality class, since the corresponding gauge models are not. It should also be

mentioned that there are other methods to make random surfaces better behaved, e.g. introduction of an extrinsic curvature term in the action (Polyakov 1986, Ambjørn *et al* 1987).

Karowsky and Thun (1985) have carried out simulations on a SA random surface model with a Euler characteristic term in the action, i.e. essentially the model governed by the action (4.5). They found that the critical point is characterised by vanishing mean curvature, $\langle \chi \rangle = 0$. They further refined their model by distinguishing between surfaces with the same χ but a different number of connected components. However, because this quantity cannot be expressed as the integral of a local density, the refined model cannot correspond to a gauge model.

We also wish to make an observation which may be relevant to the classification of three-dimensional critical phenomena. In two dimensions, the most important characteristic of a second-order phase transition is its associated conformal anomaly number, c (Cardy 1988). At least at the level of critical exponents, the O(N) model (with $-2 \leqslant N \leqslant 2$) corresponds to all real values of c between -2 and $+1$. Most of these models are not unitary. At the same time, the O(N) model is equivalent to a gas of loops, which are the only non-trivial closed manifolds that can be embedded in two dimensions, and c is related to the fugacity of these loops. Taking the models of sections 3 and 4 seriously, a natural generalisation to three dimensions is that critical phenomena be characterised by two parameters: the fugacities for loops and surfaces, corresponding to two qualitatively distinct classes of models: spin and gauge.

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